Involution requirement on a boundary makes massless fermions compactified on a finite flat disk mass protected

N.S. Mankoč Borštnik

Department of Physics, FMF, University of Ljubljana, Jadranska 19,Ljubljana, 1000

H. B. Nielsen

Department of Physics,

Niels Bohr Institute, Blegdamsvej 17,

Copenhagen, DK-2100

Abstract

The genuine Kaluza-Klein-like theories—with no fields in addition to gravity—have difficulties with the existence of massless spinors after the compactification of some space dimensions [1]. We proposed in ref. [2] such a boundary condition for spinors in $1+5$ compactified on a flat disk that ensures masslessness of spinors in $d=1+3$ as well as their chiral coupling to the corresponding background gauge gravitational field. In this paper we study the same toy model, looking this time for an involution which transforms a space of solutions of Weyl equations in $d=1+5$ from the outside of the flat disk in x^5 and x^6 into its inside, allowing massless spinor of only one handedness—and accordingly assures mass protection— and of one charge—$1/2$—and infinitely many massive spinors of the same charge. We reformulate the operator of momentum so that it is Hermitean on the vector space of spinor states obeying the involution boundary condition.

PACS numbers: Unifying theories(12.10.-g,04.60.Kz,11.15.-Ex), Kaluza-Klein-like theories(11.25.Mj,11.40.-q), mass protection mechanism(11.30.Rd), generalized Hermitean operators for momentum(03.65.-w), higher dimensional spaces(04.50.+h)

I. INTRODUCTION

The major problem of the compactification procedure in all Kaluza-Klein-like theories with only gravity and no additional gauge fields is how to ensure that massless spinors be mass protected after the compactification. Namely, even if we start with only one Weyl spinor in some even dimensional space of $d = 2$ modulo 4 dimensions (i.e. in $d = 2(2n + 1)$, $n = 0, 1, 2, \cdots$) so that there appear no Majorana mass if no conserved charges exist and families are allowed, as we have proven in ref. [3], and accordingly with the mass protection from the very beginning, a compactification of m dimensions gives rise to a spinor of one handedness in d with both handedness in $d - m$ and is accordingly not mass protected any longer.

And besides, since the spin (or the total conserved angular momentum) in the compactified part of space will in $d - m$ space manifest as a charge of a positive and a negative value and since in the second quantization procedure antiparticles of opposite charges appear anyhow, doubling the number of massless spinors when coming from $d(= 2(2n+1))$-dimensional space down to $d = 4$ and after the second quantized procedure is not in agreement with what we observe. Accordingly there must be some requirements, some boundary conditions, which ensure in a compactification procedure that only spinors of one handedness survive, if Kaluza-Klein-like theories have some meaning. However, the idea of Kaluza and Klein of having only gravity as a gauge field seems too beautiful not to have the realization in Nature.

One of us[4, 5, 6, 7, 8, 9] has for long tried to unify the spin and all the charges to only the spin, so that spinors would in $d \geq 4$ carry nothing but a spin and interact accordingly with only the gauge fields of the Poincaré group, that is with vielbeins $f^{\alpha}{}_{a}$ [13] and spin connections $\omega_{ab\alpha}$, which are the gauge fields of the Poincaré group.

In this paper we take (as we did in ref. [2]) the covariant momentum of a spinor, when applied on a spinor function ψ, to be

$$p_{0a} \;=\; f^{\alpha}{}_{a} p_{0\alpha}, \quad p_{0\alpha}\psi = p_{\alpha} - \frac{1}{2} S^{cd}\omega_{cd\alpha}. \tag{1}$$

The corresponding Lagrange density $\mathcal{L}$ for a Weyl spinor has the form $\mathcal{L} = E\frac{1}{2}[(\psi^{\dagger}\gamma^{0}\gamma^{a}p_{0a}\psi) + (\psi^{\dagger}\gamma^{0}\gamma^{a}p_{0a}\psi)^{\dagger}]$ and leads to

$$\mathcal{L} \;=\; E\psi^{\dagger}\gamma^{0}\gamma^{a}(p_{a} - \frac{1}{2}S^{cd}\omega_{cda})\psi, \tag{2}$$

with $E = \det(e^a{}_\alpha)$.

The authors of this work have tried to find a way out of this "Witten's no go theorem" for a toy model of $M^{(1+3)}\times$ a flat finite disk in $(1+5)$-dimensional space [2] by postulating a particular boundary condition, which allows a spinor to carry only one handedness after the compactification. Massless spinors then chirally couple to the corresponding background gauge gravitational field, which solves equations of motion for a free field, linear in the Riemann curvature, while the current through the wall for the massless and all the massive solutions is equal to zero.

In ref. [2] the boundary condition was written in a covariant way as

$$\hat{\mathcal{R}}\psi|_{\rm wall} \;=\; 0, \quad \hat{\mathcal{R}} = \frac{1}{2}(1 - i n^{(\rho)}{}_a n^{(\phi)}{}_b \gamma^a \gamma^b), \quad \hat{\mathcal{R}}^2 = \hat{\mathcal{R}} \tag{3}$$

with $n^{(\rho)} = (0,0,0,0,\cos\phi,\sin\phi)$, $n^{(\phi)} = (0,0,0,0,-\sin\phi,\cos\phi)$, which are the two unit vectors perpendicular and tangential to the boundary of the disk at ρ_0, respectively. The projector $\hat{\mathcal{R}}$ can for the above choice of the two vectors $n^{(\rho)}$ and $n^{(\phi)}$ be written as

$$\hat{\mathcal{R}} \;=\; \overset{56}{[-]} = \frac{1}{2}(1 - i\gamma^5\gamma^6). \tag{4}$$

The reader can find more about the Clifford algebra objects $\overset{ab}{(\pm)}, \overset{ab}{[\pm]}$ in Appendix A.

The boundary condition requires that only massless states (determined by Eq.(2)) of one (let us say right) handedness with respect to the compactified disk are allowed. Accordingly massless states of only one handedness are allowed in $d = 1+3$.

In this paper:

i) *We reformulate the boundary condition as an involution*, which transforms the solutions of the equations of motion (or their superpositions) from outside the boundary of the disk into its inside. We do this with the intention that the limitation of M^2 on a finite disk have a natural explanation, originated in a symmetry relation, allowing only *one massless state with the charge determined by the spin* in x^5, x^6 and infinitely many massive states with the same charge—so that to each mass only one state corresponds.

ii) *We redefine the definition of the momentum p^s so that it becomes Hermitean on the vector space of states fulfilling the involution boundary conditions* and we comment on the orthogonality relations of these states.

We make use of the technique presented in refs. [10, 11] when writing the equations of motion and their solutions. We briefly repeat this technique in Appendix A. It turns out that all the derivations and discussions appear to be very transparent when using this technique.

II. EQUATIONS OF MOTION AND SOLUTIONS

We assume that the two dimensional space of coordinates x^5 and x^6 is a Euclidean plane $M^{(2)}$ (with no gravity) $f^{\sigma}{}_s = \delta^{\sigma}{}_s,\ \omega_{56s} = 0$ and with the rotational symmetry around an origin.

Wave functions describing spinors in $(1+5)$-dimensional space demonstrating $M^{(1+3)}$ $\times M^{(2)}$ symmetry are required to obey the equations of motion

$$\gamma^0\gamma^a p_a \psi^{(6)} = 0, \quad a = m, s, \quad m = 0,1,2,3, \quad s = 5,6. \tag{5}$$

The most general solution for a free particle in $d = 1+5$ should be written as a superposition of all four $(2^{6/2-1})$ states of a single Weyl representation. We ask the reader to see Appendix A for the technical details on how to write a Weyl representation in terms of the Clifford algebra objects after making a choice of the Cartan subalgebra, for which we take: S^{03}, S^{12}, S^{56}. In our technique [10] one spinor representation—the four states, the eigenstates of the chosen 4Cartan subalgebra—are expressed with the following four products of projections $\stackrel{ab}{[k]}$ and nilpotents $\stackrel{ab}{(k)}$:

$$\begin{aligned}
\varphi^1_1 &= \stackrel{56}{(+)}\stackrel{03}{(+i)}\stackrel{12}{(+)}\ \psi_0, \\
\varphi^1_2 &= \stackrel{56}{(+)}\stackrel{03}{[-i]}\stackrel{12}{[-]}\ \psi_0, \\
\varphi^2_1 &= \stackrel{56}{[-]}\stackrel{03}{[-i]}\stackrel{12}{(+)}\ \psi_0, \\
\varphi^2_2 &= \stackrel{56}{[-]}\stackrel{03}{(+i)}\stackrel{12}{[-]}\ \psi_0,
\end{aligned} \tag{6}$$

where ψ_0 is a vacuum state. If we write the operators of handedness in $d = 1+5$ as $\Gamma^{(1+5)} = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\gamma^6$ $(= 2^3 i S^{03}S^{12}S^{56})$, in $d = 1+3$ as $\Gamma^{(1+3)} = -i\gamma^0\gamma^1\gamma^2\gamma^3$ $(= 2^2 i S^{03}S^{12})$ and in the two dimensional space as $\Gamma^{(2)} = i\gamma^5\gamma^6$ $(= 2S^{56})$, we find that all four states are lefthanded with respect to $\Gamma^{(1+5)}$, with the eigenvalue -1, the first two are righthanded and the second two lefthanded with respect to $\Gamma^{(2)}$, with the eigenvalues 1 and -1, respectively, while the first two are lefthanded and the second two righthanded with respect to $\Gamma^{(1+3)}$ with the eigenvalues -1 and 1, respectively.

Taking into account Eq.(6) we may write a wave function $\psi^{(6)}$ in $d=1+5$ as

$$\psi^{(6)} = (\mathcal{A} \overset{56}{(+)} + \mathcal{B} \overset{56}{[-]})\, \psi^{(4)}, \tag{7}$$

where $\mathcal{A}$ and $\mathcal{B}$ depend on x^5 and x^6, while $\psi^{(4)}$ determines the spin and the coordinate dependent part of the wave function $\psi^{(6)}$ in $d=1+3$.

Spinors which manifest masslessness in $d=1+3$ must obey the equation

$$\gamma^0 \gamma^s p_s \psi^{(6)} = 0, \quad s=5,6, \tag{8}$$

since what will demonstrate as an effective action in $d=1+3$ is

$$\begin{aligned} &\int \prod_m dx^m \mathrm{Tr}_{0123}(\int dx^5 dx^6 \mathrm{Tr}_{56}(\psi^{(6)\dagger}\gamma^0(\gamma^m p_m + \gamma^s p_s)\psi^{(6)})) = \\ &\int \prod_m dx^m \mathrm{Tr}_{0123}(\psi^{(4)\dagger}\gamma^0\gamma^m p_m \psi^{(4)}) - \int \prod_m dx^m \mathrm{Tr}_{0123}(\psi^{(4)\dagger}\gamma^0 m \psi^{(4)}), \end{aligned} \tag{9}$$

where integrals go over all the space on which the solutions are defined. Tr_{0123} and Tr_{56} mean the trace over the spin degrees of freedom in x^0, x^1, x^2, x^3 and in x^5, x^6, respectively. (One finds, for example, that $\mathrm{Tr}_{56}(\overset{56}{[\pm]}) = 1$.)

For massless spinors it must be that $\int dx^5 dx^6 \mathrm{Tr}_{56}(\psi^{(6)\dagger}\gamma^0\gamma^s p_s \psi^{(6)}) = \psi^{(4)\dagger}\gamma^0(-m)\psi^{(4)} = 0$. To find the effective action in $1+3$ for massive spinors we recognize that $\psi^{(4)\dagger}\gamma^0(-\mathcal{A}^* \overset{56}{(+)}{}^\dagger + \mathcal{B}^* \overset{56}{[-]}{}^\dagger)\gamma^s p_s(\mathcal{A} \overset{56}{(+)} + \mathcal{B} \overset{56}{[-]})\psi^{(4)} = \psi^{(4)\dagger}\gamma^0(-\mathcal{A}^* \overset{56}{(+)}{}^\dagger + \mathcal{B}^* \overset{56}{[-]}{}^\dagger)(-m)(-\mathcal{A} \overset{56}{(+)} + \mathcal{B} \overset{56}{[-]})$ $\psi^{(4)}$, with $s=5,6$, $\overset{56}{(\pm)}{}^\dagger = -\overset{56}{(\mp)}$ and $\overset{56}{[\pm]}{}^\dagger = \overset{56}{[\pm]}$, while $(^*)$ means complex conjugation. We took into account that $\gamma^0 \overset{56}{(+)} = -\overset{56}{(+)}\gamma^0$, while $\gamma^0 \overset{56}{[-]} = \overset{56}{[-]}\gamma^0$. We find that $\mathrm{Tr}_{56}(\overset{56}{(+)}{}^\dagger \overset{56}{(+)})$ $= 1$. In order that $\int dx^5 dx^6 \mathrm{Tr}_{56}(\psi^{(4)\dagger}\gamma^0(-\mathcal{A}^* \overset{56}{(+)}{}^\dagger + \mathcal{B}^* \overset{56}{[-]}{}^\dagger)\gamma^s p_s(\mathcal{A} \overset{56}{(+)} + \mathcal{B} \overset{56}{[-]})\psi^{(4)}$ will appear in $d=1+3$ as a mass term $\psi^{(4)\dagger}\gamma^0(-m)\psi^{(4)}$, we must solve the equation $\gamma^s p_s\,(\mathcal{A} \overset{56}{(+)}$ $+ \mathcal{B} \overset{56}{[-]}) = (-m)(-\mathcal{A} \overset{56}{(+)} + \mathcal{B} \overset{56}{[-]})$.

We can rewrite equations of motion in terms of the two complex superposition of x^5 and x^6, $z := x^5 + ix^6$ and $\bar{z} := x^5 - ix^6$ and their derivatives, defined as $\frac{\partial}{\partial z} := \frac{1}{2}(\frac{\partial}{\partial x^5} - i\frac{\partial}{\partial x^6})$, $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x^5} + i\frac{\partial}{\partial x^6})$ and in terms of the two projectors $\overset{56}{[\pm]} := \frac{1}{2}(1 \pm i\gamma^5\gamma^6)$ as follows

$$2i\gamma^5\{\frac{\partial}{\partial z}\overset{56}{[-]} + \frac{\partial}{\partial \bar{z}}\overset{56}{[+]}\}(\mathcal{A} \overset{56}{(+)} + \mathcal{B} \overset{56}{[-]}) = -m(-\mathcal{A} \overset{56}{(+)} + \mathcal{B} \overset{56}{[-]}). \tag{10}$$

Since in Eq.(10) $\psi^{(4)}$ would be just a spectator, we skipped it.

In the massless case the superposition of the first two states ($\psi^{(6)m=0}_{+} = \overset{56}{(+)}\,\psi^{(4)m=0}_{+}$, with $\psi^{(4)m=0}_{+} = (\alpha \overset{03}{(+i)}\overset{12}{(+)} + \beta \overset{03}{[-i]}\overset{12}{[-]})\psi_0$) or the second two states ($\psi^{(6)m=0}_{-} = \overset{56}{[-]}\,\psi^{(4)m=0}_{-}$, with

$\psi^{(4)m=0}_{-} = (\alpha \stackrel{03}{[-i]}\stackrel{12}{(+)} + \beta \stackrel{03}{(+i)}\stackrel{12}{[-]})\psi_0)$ of the left handed Weyl representation presented in Eq.(6) must be taken, with the ratio of the two parameters α and β determined by the dynamics in x^m space. In the massive case $\psi^{(6)m}$ is the superposition of all the states to which γ^5 and γ^0 separately transform the starting state: $\psi^{(6)m} = (\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]})\psi^{(4)m}_{\pm}$, with $\psi^{(4)m}_{\pm} = \{\alpha[\stackrel{03}{(+i)}\stackrel{12}{(+)} \pm \stackrel{03}{[-i]}\stackrel{12}{(+)}] + \beta[\stackrel{03}{[-i]}\stackrel{12}{[-]} \pm \stackrel{03}{(+i)}\stackrel{12}{[-]}]\}\psi_0$. The sign $\pm$ denotes the eigenvalue of γ^0 on these states.

We shall therefore simply write (as suggested in Eq.(7)) $\psi^{(6)} = (\mathcal{A} \stackrel{56}{(+)} + \mathcal{B} \stackrel{56}{[-]})\psi^{(4)}$ in the massless and the massive case, taking into account that in the massless case either $\mathcal{A}$ or $\mathcal{B}$ is nonzero, while in the massive case both are nonzero. Accordingly also $\psi^{(4)}$ differs in the massless and the massive case.

We want our states to be eigenstates of the total angular momentum operator M^{56} around a chosen origin in the flat two dimensional manifold ($M^{(2)}$)

$$M^{56} = z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}} + S^{56}. \tag{11}$$

Taking into account that $\gamma^5 \stackrel{56}{(+)} = -\stackrel{56}{[-]}$, $\gamma^5 \stackrel{56}{[-]} = \stackrel{56}{(+)}$, (see AppendixA) we end up with the equations for $\mathcal{A}$ and $\mathcal{B}$

$$\begin{aligned} \frac{\partial \mathcal{B}}{\partial z} + \frac{im}{2}\mathcal{A} = 0, \\ \frac{\partial \mathcal{A}}{\partial \bar{z}} + \frac{im}{2}\mathcal{B} = 0. \end{aligned} \tag{12}$$

For $m = 0$ we get as solutions

$$\begin{aligned} \psi^{(6)m=0}_{n+1/2} &= a_n z^n \stackrel{56}{(+)} \psi^{(4)}_{+}, \\ \psi^{(6)m=0}_{-(n+1/2)} &= b_n \bar{z}^n \stackrel{56}{[-]} \psi^{(4)}_{-},\ n \geq 0. \end{aligned} \tag{13}$$

We required $n \geq 0$ to ensure the integrability of solutions at the origin. The solutions have the eigenvalues of M^{56} equal to $(n + 1/2)$ and $-(n + 1/2)$, respectively. Having solutions of both handedness we conclude that there is no mass protection.

For $m \neq 0$ we get

$$\psi^{(6)m}_{\pm n+1/2} = a_n(J_n \stackrel{56}{(+)} - iJ_{n+1}e^{i\phi} \stackrel{56}{[-]})e^{\pm in\phi}\psi^{(4)m},\ \text{for n} \geq 0, \tag{14}$$

where J_n is the Bessel's functions of the first order. The easiest way to see that J_n and J_{n+1} determine the massive solution is to use Eq.(12), take into account that $z = \rho e^{i\phi}$, define

$r = m\rho, \rho = \sqrt{(x^5)^2 + (x^6)^2}$, recognize that $\frac{\partial}{\partial z} = \frac{1}{2} e^{-i\phi}(\frac{\partial}{\partial \rho} - \frac{i}{\rho}\frac{\partial}{\partial \phi})$ and we find $\mathcal{B} = -\frac{2}{im}\frac{\partial \mathcal{A}}{\partial \bar{z}}$. Then for the choice $\mathcal{A} = J_n e^{in\phi}$ it follows that $\mathcal{B} = -ie^{i(n+1)\phi}(\frac{n}{r} J_n - \frac{\partial J_n}{\partial r})$, which tells that $\mathcal{B} = -i J_{n+1} e^{i(n+1)\phi}$.

III. BOUNDARY CONDITIONS AND INVOLUTION

In the ref. [2] we make a choice of particular solutions of the equations of motion by requiring that $\hat{\mathcal{R}}\psi|_{\text{wall}} = 0$, Eq.(3), where the wall was put on the circle of the radius ρ_0 of the finite disk (Eqs.(3,4)). This boundary condition requires that in the massless case (since $\overset{56}{[-]}\overset{56}{(+)}= 0$ while $\overset{56}{[-]}\overset{56}{[-]}=\overset{56}{[-]}$) only the right handed solutions (Eq.13) $\psi^{(6)m=0}_{n+1/2} = a_n z^n \overset{56}{(+)}$ $\psi^{(4)m=0}_{+}$ (that is the left handed with respect to $SO(1,3)$) are allowed, while the left handed solutions must be zero ($b_n = 0$) making the mass protection mechanism work in $d = 1 + 3$. This boundary condition allows all the angular momenta $M^{56} = 1/2, 3/2, \cdots$, which is still not what we would expect to have as a charge of massless spinors in $1 + 3$, namely $n = 1/2$ only.

In the massive case the boundary condition determines masses of solutions, since only the solutions with $J_{n+1}|_{\rho=\rho_0} = 0$ are allowed from the same reason as discussed for the massless case. This boundary condition determines masses of spinors through the relation $m_{in+1/2}\rho_0$ is equal to a i^{th} zero of J_{n+1} ($J_{n+1}(m_{in+1/2}\rho_0) = 0$). In the massive case all the zeros of any $J_{n+1}(m_{in+1/2}\rho_0) = 0$ contribute.

This time we look for the *involution boundary conditions.* First we recognize that for a flat M^2 the Z_2 or involution symmetry can be recognized: *The transformation* $\rho/\rho_0 \to \frac{\rho_0}{\rho}$ (which can be written also as $z/\rho_0 \to \frac{\rho_0}{\bar{z}}$) *transforms the exterior of the disk into the interior of the disk* and conversely.

Then we extend the involution operator to operate also on the space of solutions

$$\begin{aligned} \hat{\mathcal{O}} &= (I - 2\hat{\mathcal{R}}')|_{z/\rho_0 \to \rho_0/\bar{z}}, \\ \hat{\mathcal{O}}^2 &= I. \end{aligned} \tag{15}$$

The involution condition $\hat{\mathcal{O}}^2 = I$ requires, that $\hat{\mathcal{R}}'$ is a projector

$$(\hat{\mathcal{R}}')^2 = \hat{\mathcal{R}}' \tag{16}$$

and can be written as $\hat{\mathcal{R}}' = \hat{\mathcal{R}} + \hat{\mathcal{R}}_{add}$, where $\hat{\mathcal{R}}_{add}$ must be a nilpotent operator fulfilling the conditions

$$(\hat{\mathcal{R}}_{add})^2 = 0, \quad \hat{\mathcal{R}}_{add}\hat{\mathcal{R}} = 0, \quad \hat{\mathcal{R}}\hat{\mathcal{R}}_{add} = \hat{\mathcal{R}}_{add}, \tag{17}$$

We had $\hat{\mathcal{R}} = \stackrel{56}{[-]}$, which is the projector. Since we find that $\stackrel{56}{[-]}\stackrel{56}{(-)} = \stackrel{56}{(-)}$ (see Appendix A), while $\stackrel{56}{(-)}\stackrel{56}{[-]} = 0$, we can choose $\hat{\mathcal{R}}_{add} = \alpha \stackrel{56}{(-)}$, where α is any function of z and $\frac{\partial}{\partial z}$. Let us point out that $\hat{\mathcal{R}}_{add}$ is not a Hermitean operator, since $\stackrel{56}{(-)}^\dagger = -\stackrel{56}{(+)}$ and $z^\dagger = \bar{z}, (\frac{\partial}{\partial z})^\dagger = \frac{\partial}{\partial \bar{z}}$. Accordingly also neither $\hat{\mathcal{R}}'$ nor $\hat{\mathcal{O}}$ is a Hermitean operator.

We now make a choice of a natural boundary conditions on the wall $\rho = \rho_0$

$$\{\hat{\mathcal{O}}\psi \;=\; \psi\}|_{\text{wall}}, \tag{18}$$

saying that what ever the involution operator is, the state ψ and its involution $\hat{\mathcal{O}}\psi$ must be the same on the wall, that is at $\rho = \rho_0$.

It is worthwhile to write the involution operator $\hat{\mathcal{O}}$ and correspondingly the projector $\hat{\mathcal{R}}'$ in a covariant way. Recognizing that ${n^{(\rho)}}_a \gamma^a {n^{(\rho)}}_b p^b = i\{(e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}) \stackrel{56}{(-)} + (\frac{\partial}{\partial z} + e^{-2i\phi}\frac{\partial}{\partial \bar{z}}) \stackrel{56}{(+)}\}$ we may write

$$\begin{aligned}\hat{\mathcal{R}}' &= \frac{1}{2}(1 - i{n^{(\rho)}}_a {n^{(\phi)}}_b \gamma^a\gamma^b)(1 - \beta {n^{(\rho)}}_a \gamma^a {n^{(\rho)}}_b p^b) \\ &= \stackrel{56}{[-]}\,(I - \beta i e^{i\phi}\frac{\partial}{\partial \rho} \stackrel{56}{(-)}).\end{aligned} \tag{19}$$

This is just our generalized projector $\hat{\mathcal{R}}'$, if we make a choice for α from Eq.(17) as follows: $\alpha = -\beta i(e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}})$(since $\stackrel{56}{[-]}\stackrel{56}{(-)} = \stackrel{56}{(-)}$), while $\stackrel{56}{[-]}\stackrel{56}{(+)} = 0$), where β is any complex number.

The projector $\hat{\mathcal{R}}'$ (Eq.(19)) entering into the orbifolding condition $(\hat{\mathcal{O}}\psi)|_{wall} = (\psi)|_{wall}$, with $\hat{\mathcal{O}} = I - 2\hat{\mathcal{R}}'$, requires that

$$(\stackrel{56}{[-]}\,\{I - i\beta e^{i\phi}\frac{\partial}{\partial \rho} \stackrel{56}{(-)}\}\psi)|_{wall} = 0. \tag{20}$$

For the massless case we obtain: $(\stackrel{56}{[-]}\,[I - i\beta e^{i\phi}\frac{\partial}{\partial \rho} \stackrel{56}{(-)}]z^n \stackrel{56}{(+)})|_{wall} = (-i\beta e^{2i\phi} n z^{n-1} \stackrel{56}{[-]}\,)|_{wall} = 0$, *which has the solution for an arbitrarily chosen β only if $n = 0$.*

For the massive case we have: $(\stackrel{56}{[-]}\,\{I - i\beta e^{i\phi}\frac{\partial}{\partial \rho} \stackrel{56}{(-)}\}(J_n \stackrel{56}{(+)} - iJ_{n+1}e^{i\phi} \stackrel{56}{[-]})e^{in\phi})|_{wall} = ((-iJ_{n+1} + i\beta\frac{\partial J_n}{\partial \rho})e^{i(n+1)\phi} \stackrel{56}{[-]})|_{wall} = 0$, which has again *the solution for an arbitrarily chosen β only for $n = 0$.* In this case we namely have $J_1 = -\frac{\partial J_0}{\partial m_{im+1/2}\rho}$. If we chose that $J_1|_{wall} = 0$, then $\frac{\partial J_0}{\partial m_{in+1/2}\rho}|_{wall} = 0$ and the relation $(-J_1 + \beta\frac{\partial J_0}{\partial \rho})|_{wall} = 0$ for any β. This relation is fulfilled for infinite many masses $m_{i1/2}, i = 1, \cdots$, where index i counts the zeros of J_1 determined by the relation $J_1(m_{i1/2}\rho)|_{wall} = 0$.

We conclude this section by recognizing that we have on the disk *only one massless solution*, namely

$$\psi^{m=0}_{1/2} = a_0 \stackrel{56}{(+)} \tag{21}$$

infinite many massive solutions, namely

$$\psi^{m^i}_{1/2} = a_i(J_{oi}(\alpha_{1i}\rho/\rho_0) \stackrel{56}{(+)} - iJ_{1i}(\alpha_{1i}\rho/\rho_0)), \tag{22}$$

with i which denotes the i-th zero of $J_{\alpha_{1i}} = 0$. All the solutions have the eigenvalue of M^{56} equal to $1/2$ and obey the equations of motion (Eq.(12)) *and the orbifolding boundary condition* (Eq.(18)) with $\hat{\mathcal{R}}'$ from Eq.(19). We get the corresponding solutions in $d = 1 + 5$ ($\psi^{(6)}$) by multiplying the wave functions of Eqs.(21,22) with the corresponding $\psi^{(4)}$, which distinguish among the massless and the massive solutions as described in Sect.5.

IV. CURRENT THROUGH THE WALL

The current perpendicular to the wall can be written as

$$n^{(\rho)s} j_s \;=\; \psi^\dagger \gamma^0 \gamma^s n^{(\rho)}_s \psi = \psi^\dagger \hat{j}_\perp \psi, \quad \hat{j}_\perp = -\gamma^0 \{e^{-i\phi} \stackrel{56}{(+)} + e^{i\phi} \stackrel{56}{(-)}\}. \tag{23}$$

We need to know the current through the wall, which for physically acceptable cases when spinors are localized inside the disk (involution transforms outside the disk into its inside, or equivalently, it transforms inside the disk into its outside) must be zero. We accordingly expect that the current through the wall is equal to zero:

$$\{\psi^\dagger \hat{j}_\perp \psi\}|_{\rm wall} = 0 = \{\psi^\dagger \hat{\mathcal{O}}^\dagger \hat{j}_\perp \hat{\mathcal{O}} \psi\}|_{\rm wall}. \tag{24}$$

Since $\hat{\mathcal{O}}^\dagger = I - 2(\hat{\mathcal{R}} + \hat{\mathcal{R}}^\dagger_{add})$ and $\hat{\mathcal{R}}^\dagger_{add} = (-i\beta e^{i\phi} \frac{\partial}{\partial\rho} \stackrel{56}{(-)})^\dagger = -i\beta^* e^{-i\phi} \frac{\partial}{\partial\rho} \stackrel{56}{(+)}$, it follows that $\hat{\mathcal{O}}^\dagger \hat{j}_\perp \hat{\mathcal{O}} = -\hat{j}_\perp - 2i\gamma^0 \frac{\partial}{\partial\rho}(\beta^* \stackrel{56}{[-]} - \beta \stackrel{56}{[+]})$.

It must then be

$$\{\psi^\dagger \hat{j}_\perp \psi\}|_{\rm wall} = 0 = -(\psi^\dagger \{\hat{j}_\perp + 2i\gamma^0 \frac{\partial}{\partial\rho}(\beta^* \stackrel{56}{[-]} - \beta \stackrel{56}{[+]})\}\psi)|_{\rm wall} \tag{25}$$

for any β and any superposition of the states obeying our involution boundary condition of Eq.(18), with $\hat{\mathcal{O}}$ (Eq.15) and $\hat{\mathcal{R}}'$ (Eq.19). One easily finds that the current through the wall

(Eq.(25)) is equal to zero for the massless and all the massive solutions (or any superposition of solutions) obeying the involution boundary condition (Eq.(18)).

V. HERMITICITY OF THE OPERATORS AND ORTHOGONALITY OF SOLUTIONS

In this section we redefine the operators p_s and $(\gamma^s p_s)^2$ so that they become Hermitean on the vector space of the states fulfilling our particular involution boundary conditions (Eqs.(15,19)), that is on the vector space of the massless state $a_0 \stackrel{56}{(+)}$ (Eq.21) and the massive states $a_i(J_{0i} \stackrel{56}{(+)} -iJ_{1i} \stackrel{56}{[-]} e^{i\phi})$ (Eq.(22)), where i counts the $i-$th zero α_{1i} of J_{1i}. All the states have the eigenvalue of M^{56} equal to 1/2. We also comment on the orthogonality properties of these states.

We expect that on the space of all the states fulfilling our involution boundary conditions
i) the operators p_s are Hermitean,
ii) the states are accordingly orthogonal ($\int d^2x\psi_i^{\dagger}(\gamma^s p_s)^2)\psi_j = \int d^2x\psi_i^{\dagger}\psi_j m^2\delta_{ij}$).

First we recognize that the operator p_s is not Hermitean on the states, which are not zero on the wall (that is at $\rho = \rho_0$) since for those state $\int d^2x\psi_i^{\dagger} p_s\psi_j + \int d^2x(-p_s\psi_i)^{\dagger}\psi_j \neq 0$.

We therefore replace p_s with $\hat{p}_s$

$$\hat{p}_s = i\{\frac{\partial}{\partial x^s} - \frac{1}{2}\frac{x^s}{\rho}\delta(\rho-\rho_0) \stackrel{56}{[+]}\}. \tag{26}$$

One finds, for example, that $\mathrm{Tr}_{56}\int d^2x\psi_j^{\dagger}(\hat{p}_s\psi_i) = \mathrm{Tr}_{56}\int d^2x(\hat{p}_s\psi_j)^{\dagger}\psi_i = -i\pi/2\rho_0^{2(n+1)}\varepsilon$, with $\varepsilon = 1, -i$ for $s = 5,6$, respectively when $\psi_i = \rho^n e^{in\phi} \stackrel{56}{(+)}$ and $\psi_j = \rho^{n+1}e^{i(n+1)\phi} \stackrel{56}{(+)}$.

Since $\stackrel{56}{(+)}$ (the massless state) and $\stackrel{56}{[-]}$ are orthogonal in the spin part, while the matrix element of $\hat{p}_s$ is zero between the massless state and $J_{0i} \stackrel{56}{(+)}$, we check instead the hermiticity properties of the operator $(\gamma^s\hat{p}_s)^2$

$$\begin{aligned}
\gamma^s\hat{p}_s\gamma^t\hat{p}_t &= p_s p^s \\
&+ \frac{1}{2}\{[\frac{\partial}{\partial\rho}\delta(\rho-\rho_0) + \frac{1}{\rho}\delta(\rho-\rho_0) + \delta(\rho-\rho_0)(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi})] \stackrel{56}{[+]} \\
&+ \delta(\rho-\rho_0)(\frac{\partial}{\partial\rho} - \frac{i}{\rho}\frac{\partial}{\partial\phi}) \stackrel{56}{[-]}\},
\end{aligned} \tag{27}$$

where $p_s p^s = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\rho}\frac{\partial}{\partial\phi}$.

Let us first check the Hermiticity and accordingly the orthogonality relations between the massless and any of the massive states obeying our involution boundary condition. Since the spinor parts $\stackrel{56}{(+)}$ and $\stackrel{56}{[-]}$ are orthogonal, we only have to check the Hermiticity properties with respect to the $\stackrel{56}{(+)}$ component of the massive states. We find that

$$\begin{aligned}
&\int d^2x \mathrm{Tr}_{56}(J_{0i} \stackrel{56}{(+)})^\dagger[(\gamma^s \hat{p}_s)^2 \stackrel{56}{(+)}] = \pi \frac{\alpha_{1i}}{\rho_0}(\rho J_{1i})|_{\rho=\rho_0} = 0 \\
&= \int d^2x \mathrm{Tr}_{56}[(\gamma^s \hat{p}_s)^2 J_{0i} \stackrel{56}{(+)}]^\dagger \stackrel{56}{(+)} = \pi(-\frac{\alpha_{1i}}{\rho_0}\rho J_{1i})|_{\rho=\rho_0},
\end{aligned} \tag{28}$$

due to the properties of the Bessel functions $J_{1i} = -\frac{\rho_0}{\alpha_{1i}}\frac{\partial J_{0i}}{\partial \rho}$ with $J_{1i}(\alpha_{1i}) = 0$ (due to our particular boundary condition), and of the delta function $\int_0^{\rho_0} f(\rho)\frac{\partial \delta(\rho-\rho_0)}{\partial \rho} = -\frac{\partial f}{\partial \rho}$, while $m_i = \frac{\alpha_{1i}}{\rho_0}$. Accordingly the massless state is orthogonal to all the massive states.

We also find that the operator $(\gamma^s \hat{p}_s)^2$ is Hermitean on the space of massive states. Taking into account that for the Bessel functions $\frac{\partial J_{1i}}{\partial \rho} = \frac{\alpha_{1i}}{\rho_0} J_{0i} + \frac{\rho_0}{\alpha_{1i}}\frac{1}{\rho}\frac{\partial J_{0i}}{\partial \rho}$ one finds that for $i \neq k$ it follows

$$\begin{aligned}
&\int d^2x \mathrm{Tr}_{56}(J_{0i} \stackrel{56}{(+)} -iJ_{1i} \stackrel{56}{[-]} e^{i\phi})^\dagger[(\gamma^s \hat{p}_s)^2(J_{0k} \stackrel{56}{(+)} -iJ_{1k} \stackrel{56}{[-]} e^{i\phi})] = \\
&2\pi\{\int_0^{\rho_0} \rho d\rho[-(m_k)^2(J_{0i}J_{0k} + J_{1i}J_{1k})] + \frac{1}{2}(-\rho\frac{\partial J_{0i}}{\partial \rho} J_{0k} + \rho J_{1i} J_{0k}\frac{\alpha_{1k}}{\rho_0})|_{\rho=\rho_0}\} = \\
&2\pi(\rho J_{0i}J_{1k} + \rho J_{1i}J_{0k})|_{\rho=\rho_0} = 0,
\end{aligned} \tag{29}$$

since $J_{1k}(\alpha_{1k}) = 0$. We checked accordingly the Hermiticity of the operator $(\gamma^s \hat{p}_s)^2$ on the vector space of the massive states and correspondingly also the orthogonality of these states.

Let us add that

$$\begin{aligned}
&\int d^2x \mathrm{Tr}_{56}(J_{0i} \stackrel{56}{(+)} -iJ_{1i} \stackrel{56}{[-]} e^{i\phi})^\dagger[(\gamma^s \hat{p}_s]^2(J_{0i} \stackrel{56}{(+)} -iJ_{1i} \stackrel{56}{[-]} e^{i\phi}) = \\
&-(m_k)^2\pi(\rho^2(J^2_{0i} + J^2_{1i}))|_{\rho=\rho_0} = \pi(\rho^2 J^2_{0i})|_{\rho=\rho_0}, \\
&\int d^2x \mathrm{Tr}_{56}((\stackrel{56}{+}))^\dagger[(\gamma^s \hat{p}_s]^2 \stackrel{56}{(+)} = 0.
\end{aligned} \tag{30}$$

So we conclude that on the vector space of all the states, which obey our particular boundary condition (Eqs.(15,19)) the operator $(\gamma^s \hat{p}_s)^2$ (Eqs.(26,27)) is Hermitean and the states obeying also the equations of motion (Eqs.(21,22), are orthogonal.

Looking at Eq.(30), we recognize, that $m_i = \frac{\alpha_{0i}}{\rho_0}$ determine the masses of the states $\psi^{(6)m}_{1/2i}$. We read in Eq.(30) the normalization factor of the massive states, while the massless state has to be normalized according to the relation $\int d^2x \mathrm{Tr}_{56}((\stackrel{56}{+}))^\dagger \stackrel{56}{(+)} = 2\pi\frac{\rho_0^2}{2}$.

VI. PROPERTIES OF SPINORS IN $d = 1 + 3$

To study how do spinors couple to the Kaluza-Klein gauge fields in the case of $M^{(1+5)}$, "broken" to $M^{(1+3)}\times$ a flat disk with ρ_0 and with the involution boundary condition, which allows only right handed spinors at ρ_0, we first look for (background) gauge gravitational fields, which preserve the rotational symmetry on the disk. Following ref. [2] we find for the background vielbein field

$$e^a{}_\alpha = \begin{pmatrix} \delta^m{}_\mu & e^m{}_\sigma = 0 \\ e^s{}_\mu & e^s{}_\sigma \end{pmatrix}, f^\alpha{}_a = \begin{pmatrix} \delta^\mu{}_m & f^\sigma{}_m \\ 0 = f^\mu{}_s & f^\sigma{}_s \end{pmatrix}, \tag{31}$$

with $f^\sigma{}_m = A_\mu \delta^\mu{}_m \varepsilon^\sigma{}_\tau x^\tau$ and the spin connection field

$$\omega_{st\mu} = -\varepsilon_{st} A_\mu, \quad \omega_{sm\mu} = -\frac{1}{2} F_{\mu\nu} \delta^\nu{}_m \varepsilon_{s\sigma} x^\sigma. \tag{32}$$

The $U(1)$ gauge field A_μ depends only on x^μ. All the other components of the spin connection fields are zero, since for simplicity we allow no gravity in $(1+3)$ dimensional space.

To determine the current, coupled to the Kaluza-Klein gauge fields A_μ, we analyze the spinor action

$$\begin{aligned} \mathcal{S} \;=\; & \int d^dx E \bar{\psi}^{(6)} \gamma^a p_{0a} \psi^{(6)} = \int d^dx \bar{\psi}^{(6)} \gamma^m \delta^\mu{}_m p_\mu \psi^{(6)} + \\ & \int d^dx \bar{\psi}^{(6)} \gamma^m (-) S^{sm} \omega_{sm\mu} \psi^{(6)} + \int d^dx \bar{\psi}^{(6)} \gamma^s \delta^\sigma{}_s p_\sigma \psi^{(6)} + \\ & \int d^dx \bar{\psi}^{(6)} \gamma^m \delta^\mu{}_m A_\mu (\varepsilon^\sigma{}_\tau x^\tau p_\sigma + S^{56}) \psi^{(6)}. \end{aligned} \tag{33}$$

$\psi^{(6)}$ are solutions of the Weyl equation in $d = 1+3$. E is for $f^\alpha{}_a$ from (31) equal to 1. The first term on the right hand side of Eq.(33) is the kinetic term (together with the last term defines the covariant derivative $p_{0\mu}$ in $d = 1+3$). The second term on the right hand side contributes nothing when integration over the disk is performed, since it is proportional to x^σ ($\omega_{sm\mu} = -\frac{1}{2} F_{\mu\nu} \delta^\nu{}_m \varepsilon_{s\sigma} x^\sigma$).

We end up with

$$j^\mu = \int d^2x \bar{\psi}^{(6)} \gamma^m \delta^\mu{}_m M^{56} \psi^{(6)} \tag{34}$$

as the current in $d = 1+3$. The charge in $d = 1+3$ is proportional to the total angular momentum $M^{56} = L^{56} + S^{56}$ on a disk, for either massless or massive spinors.

VII. CONCLUSIONS

In this paper we were looking for what we call a "natural boundary condition"—a condition which would, due to some symmetry relations, make massless spinors which live in M^{1+5} and carry nothing but the charge to live in $M^{(1+3)}\times$ a flat disk, manifesting in $M^{(1+3)}$, if massless, as a left handed spinor (with no right handed partner) and would accordingly be mass protected. The spin in x^5 and x^6 of the left handed massless spinor should in $M^{(1+3)}$ manifest as the charge and should chirally couple with the Kaluza-Klein type of charge of only one value to the corresponding gauge field. The last requirement ensures that after the second quantization procedure a particle and an antiparticle would not appear each of $\pm$ of the particular charge.

We found the involution boundary condition

$$\begin{aligned} \{\hat{\mathcal{O}}\psi \;&=\; \psi\}|_{\rm wall}, \quad \hat{\mathcal{O}} = I - 2\hat{\mathcal{R}}', \\ \hat{\mathcal{R}}' \;&=\; \stackrel{56}{[-]}\, (I + \beta i [e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}]\, \stackrel{56}{(-)}), \end{aligned} \tag{35}$$

where β is any complex number and can be written in a covariant way as

$$\hat{\mathcal{R}}' = \frac{1}{2}(1 - i n^{(\rho)}{}_a\, n^{(\phi)}{}_b\, \gamma^a \gamma^b)(1 - \beta n^{(\rho)}{}_a\, \gamma^a\, n^{(\rho)}{}_b\, p^b) = \stackrel{56}{[-]}\, (I + \beta i [e^{2i\phi}\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}]\, \stackrel{56}{(-)}). \tag{36}$$

$\hat{\mathcal{O}}$ transforms solutions of the Weyl equations inside the flat disk into outside of it (or conversely) and allows in the massless case only the right handed spinor to live on the disk and accordingly manifests left handedness in $M^{(1+3)}$. The massless and the massive solutions carry in the fifth and sixth dimension (only) the spin 1/2, which then manifests as the charge in $d = 1+3$. The massless solution is mass protected.

We defined a generalized momentum p_s

$$\hat{p}_s = i\{\frac{\partial}{\partial x^s} - \frac{1}{2}\begin{pmatrix}\cos\phi \\ \sin\phi\end{pmatrix} \delta(\rho - \rho_0)\, \stackrel{56}{[+]}\}, \tag{37}$$

which is Hermitean on the vector space of states obeying our involution boundary condition (Eqs.(35,15,19)).

The operator $\gamma^s \hat{p}_s \gamma^t \hat{p}_t$ is Heritean on the vector space of states obeying our boundary condition and the solutions of the equations of motion—the massless one (Eq.(21) and the massive ones (Eq.(22)—are accordingly orthogonal, with the eigen values of this operator which demonstrate the masses of states.

The negative $-1/2$ charge states appear only after the second quantization procedure in agreement with what we observe.

VIII. ACKNOWLEDGEMENT

One of the authors (N.S.M.B.) would like to warmly thank Jože Vrabec for his very fruitful discussions, which help a lot to clarify the concept of involution and to use it in the right way.

APPENDIX A: SPINOR REPRESENTATION TECHNIQUE IN TERMS OF CLIFFORD ALGEBRA OBJECTS

We define[10] spinor representations as superposition of products of the Clifford algebra objects γ^a so that they are eigen states of the chosen Cartan sub algebra of the Lorentz algebra $SO(d)$, determined by the generators $S^{ab} = i/4(\gamma^a\gamma^b - \gamma^b\gamma^a)$. By introducing the notation

$$\begin{aligned}
\overset{ab}{(\pm i)}: &= \frac{1}{2}(\gamma^a \mp \gamma^b), \quad \overset{ab}{[\pm i]}:= \frac{1}{2}(1 \pm \gamma^a\gamma^b), \text{ for } \eta^{aa}\eta^{bb} = -1, \\
\overset{ab}{(\pm)}: &= \frac{1}{2}(\gamma^a \pm i\gamma^b), \quad \overset{ab}{[\pm]}:= \frac{1}{2}(1 \pm i\gamma^a\gamma^b), \text{ for } \eta^{aa}\eta^{bb} = 1,
\end{aligned} \tag{A1}$$

it can be checked that the above binomials are really "eigenvectors" of the generators S^{ab}

$$S^{ab}\, \overset{ab}{(k)}: = \frac{k}{2}\, \overset{ab}{(k)}, \quad S^{ab}\, \overset{ab}{[k]}:= \frac{k}{2}\, \overset{ab}{[k]}\,. \tag{A2}$$

Accordingly we have

$$\begin{aligned}
\overset{03}{(\pm i)}: &= \frac{1}{2}(\gamma^0 \mp \gamma^3), \quad \overset{03}{[\pm i]}:= \frac{1}{2}(1 \pm \gamma^0\gamma^3), \\
\overset{12}{(\pm)}: &= \frac{1}{2}(\gamma^1 \pm i\gamma^2), \quad \overset{12}{[\pm]}:= \frac{1}{2}(1 \pm i\gamma^1\gamma^2), \\
\overset{56}{(\pm)}: &= \frac{1}{2}(\gamma^5 \pm i\gamma^6), \quad \overset{56}{[\pm]}:= \frac{1}{2}(1 \pm i\gamma^5\gamma^6),
\end{aligned} \tag{A3}$$

with eigenvalues of S^{03} equal to $\pm\frac{i}{2}$ for $\overset{03}{(\pm i)}$ and $\overset{03}{[\pm i]}$, and to $\pm\frac{1}{2}$ for $\overset{12}{(\pm)}$ and $\overset{12}{[\pm]}$, as well as for for $\overset{56}{(\pm)}$ and $\overset{56}{[\pm]}$.

We further find

$$
\begin{aligned}
\gamma^a \overset{ab}{(k)} &= \eta^{aa} \overset{ab}{[-k]}, \quad \gamma^b \overset{ab}{(k)}= -ik \overset{ab}{[-k]},\\
\gamma^a \overset{ab}{[k]} &= \overset{ab}{(-k)}, \qquad\quad \gamma^b \overset{ab}{[k]}= -ik\eta^{aa} \overset{ab}{(-k)} \,.
\end{aligned}
\tag{A4}
$$

We also find

$$
\begin{aligned}
&\overset{ab}{(k)}\overset{ab}{(k)}= 0, \quad \overset{ab}{(k)}\overset{ab}{(-k)}= \eta^{aa} \overset{ab}{[k]}, \quad \overset{ab}{[k]}\overset{ab}{[k]}=\overset{ab}{[k]}, \qquad \overset{ab}{[k]}\overset{ab}{[-k]}= 0,\\
&\overset{ab}{(k)}\overset{ab}{[k]}= 0, \quad \overset{ab}{[k]}\overset{ab}{(k)}=\overset{ab}{(k)}, \qquad \overset{ab}{(k)}\overset{ab}{[-k]}=\overset{ab}{(k)}, \qquad \overset{ab}{[k]}\overset{ab}{(-k)}= 0.
\end{aligned}
\tag{A5}
$$

To represent one Weyl spinor in $d = 1 + 5$, one must make a choice of the operators belonging to the Cartan sub algebra of 3 elements of the group $SO(1, 5)$

$$
S^{03}, S^{12}, S^{56}. \tag{A6}
$$

Any eigenstate of the Cartan sub algebra (Eq.(A6)) must be a product of three binomials, each of which is an eigenstate of one of the three elements. A left handed spinor ($\Gamma^{(1+5)} = -1$) representation with $2^{6/2-1}$ basic states is presented in Eq.(6). for example, the state $\overset{03}{(+i)}\overset{12}{(+)}\overset{56}{(+)}\;\psi_0$, where ψ_0 is a vacuum state (any, which is not annihilated by the operator in front of the state) has the eigenvalues of S^{03}, S^{12} and S^{56} equal to $\frac{i}{2}$, $\frac{1}{2}$ and $\frac{1}{2}$, correspondingly. All the other states of one representation of $SO(1, 5)$ follow from this one by just the application of all possible $S^(ab)$, which do not belong to Cartan sub algebra.

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[13] ${f^{\alpha}}_a$ are inverted vielbeins to ${e^a}_\alpha$ with the properties ${e^a}_\alpha f^\alpha_b = {\delta^a}_b,\ e^a_\alpha {f^\beta}_a = \delta^\beta_\alpha$. Latin indices $a, b, .., m, n, .., s, t, ..$ denote a tangent space (a flat index), while Greek indices $\alpha, \beta, .., \mu, \nu, ..\sigma, \tau..$ denote an Einstein index (a curved index). Letters from the beginning of both the alphabets indicate a general index ($a, b, c, ..$ and $\alpha, \beta, \gamma, ..$), from the middle of both the alphabets the observed dimensions $0, 1, 2, 3$ ($m, n, ..$ and $\mu, \nu, ..$), indices from the bottom of the alphabets indicate the compactified dimensions ($s, t, ..$ and $\sigma, \tau, ..$). We assume the signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$.